International Journal of Heat and Mass Transfer 52 (2009) 1913–1917

Contents lists available at ScienceDirect

International Journal of Heat and Mass Transfer

journal homepage: www.elsevier.com/locate/ijhmt

Numerical solution of a moving-boundary problem with variable latent heat

Rajeev, K.N. Rai, S. Das *

Department of Applied Mathematics, Institute of Technology, Banaras Hindu University, Varanasi 221 005, India

ARTICLE INFO

Article history:
Received 21 January 2008
Received in revised form 17 May 2008
Available online 26 December 2008

Keywords:
Moving-boundary problem
Sediment transport
Shoreline problem
Chebyshev polynomial

ABSTRACT

The problem that arises during the movement of the shoreline in a sedimentary ocean basin is a moving-boundary problem with variable latent heat. A numerical method is presented for the solution of this problem. The differential equations governing the above process are converted into initial value problem of vector–matrix form. The time function is approximated by Chebyshev series and the operational matrix of integration is applied. The solution of the problem is then found in terms of Chebyshev polynomials of the second kind. The solution is utilized iteratively in the interface equation to determine time taken to attain a given shoreline position. The numerical results are obtained using Mathematica software and are compared graphically with the values obtained from a published analytical solution.

1. Introduction

An interesting moving-boundary problem, related to the movement of the shoreline in a sedimentary ocean basin (a shoreline problem), occurs in the field of geological science. In 2000, Swenson et al. [3] utilized an analogy with one-phase melting and developed a mathematical model for movement of shoreline in a sedimentary basin in response to changes in sediment line flux, tectonic subsidence of earth's crust and sea level change. This problem is different from the standard one-phase melting and freezing problems because in shoreline problem, latent heat is a function of space and time. Recently, an analytical similarity solution of a Stefan problem with variable latent heat (a limit case of the shoreline model) has been presented by Voller et al. [4]. Later, Capart et al. [12] has presented some important and practical generalizations of the analytical solution presented by Voller et al. [4]. Analytical solutions of moving-boundary problems are difficult to obtain except for a limited number of special cases [1,2].

Due to difficulties in obtaining analytical solutions, there has been extensive development of numerical methods which in many cases more practical in solving moving-boundary problems. There are three main numerical approaches of solving the moving-boundary problems. The first is fixed grid methods [5,6] where a grid of nodes remain fixed in space and the boundary is tracked by use of an auxiliary variable, e.g., the enthalpy method. The second approach is deforming grid methods [3,7,8], in which a line of node is located on the moving-boundary and, as the solution evolves, the space grid deforms to ensure that these nodes remain on the boundary. The last approach is hybrid methods [10] which employ element of both fixed and deforming grids, e.g., local front tracking.

In 2006, Voller et al. [13] has presented a novel moving-boundary problem related to shoreline movement in a sedimentary basin, which was solved by enthalpy method. In this article he has shown how shoreline problem can be solved by using the same numerical tools which were already used for solving classical stefan's melting problem.

The objective of this paper is to present a numerical solution of a moving-boundary problem with variable latent heat [4]. The differential equations governing the process (sediment transport and deposition) are converted into an initial value problem of vector–matrix form. The time function is approximated by Chebyshev series of the second kind and the operational matrix of integration is applied [11] on it. The solution of initial value problem is utilized iteratively in the interface condition to determine the time taken for a given shoreline position. A brief sensitivity study is also performed.

2. The shoreline problem

Shoreline problem involves the shoreline propagation in a sedimentary basin due to a sediment line flux, tectonic subsidence of the earth's crust, and sea level change. A schematic cross section of such a basin indicating the variables is shown in Fig. 1 (Voller et al. [4]).

The governing differential equations for the sediment transport and deposition in a sub-aerial, fluvial domain (a net depositional river basin) is given by the diffusion equation [3,4,9].

$$\frac{\partial \eta}{\partial t} = \nu \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial b}{\partial t}, \quad 0 \leqslant x \leqslant s(t) \tag{1}$$

* Corresponding author. Tel./fax: +91 0542 230 7035.
E-mail address: subir_das08@hotmail.com (S. Das).

0017-9310/$ - see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijheatmasstransfer.2008.08.036

Nomenclature

η	height of sediment above datum, m
t	time
x	space variable, m
b	height of the earth's crust (basement) above datum, m
$z(t)$	ocean level above datum, m
$s(t)$	shoreline position, m
$u(t)$	position of intersection between off-shore sediment wedge and basement, m
$\bar{q}$	Prescribed sediment line flux $m^3m^{-1}t^{-1}$
γ	a constant
α	slope of off-shore sediment wedge
β	slope of basement
ν	diffusion coefficient, m^2t^{-1}
τ	generalized time
k	step length, m
$F(t)$	Chebyshev vector
f_j	Chebyshev polynomial of second kind
T	transpose of matrix

where $\eta(x,t)$ is the height of the sediment above a datum, ν is the effective fluvial diffusivity, which depends primarily on the water discharge in the river system, and b is height of earth's crust. The boundary conditions on (1) are

$$\nu\frac{\partial\eta}{\partial x}\bigg|_{x=0} = -\bar{q}(t) \quad \text{and} \quad \eta(s,t) = z(t) \tag{2}$$

where $\bar{q}$ is a prescribed sediment line flux and $z(t)$ is the ocean level above the datum.

A condition for the advance or retreat of the shoreline in an off-shore submarine domain [4,6] is given by

$$-\nu\frac{\partial\eta}{\partial x}\bigg|_{S(t)} = (u-s)\left[\alpha\frac{ds}{dt}+\frac{dz}{dt}\right] - \int_s^u \frac{\partial b}{\partial t}dx \tag{3}$$

with the initial condition $\quad s(0) = 0 \quad$ (4)

where α is slope of the off-shore sediment wedge and $u(t)$ is the lateral position where the toe of the submarine sediment wedge intersects the ocean basement.

A specific case [4] for above shoreline model involves a shoreline problem with a fixed line flux, a constant ocean level ($z = 0$), no tectonic subsidence of the earth's crust, and a constant sloping basement $\beta < \alpha$. This scenario is a reasonable approximation for some modern continental margins. Under this specific case the governing equation, boundary conditions and initial condition (1)–(4) reduce to following one-phase moving-boundary problem with variable latent heat

$$\frac{\partial\eta}{\partial t} = \nu\frac{\partial^2\eta}{\partial x^2}, \quad 0 \leqslant x \leqslant s(t) \tag{5}$$

with initial and boundary conditions

$$\eta(x,0) = 0, \nu\frac{\partial\eta}{\partial x}\bigg|_{x=0} = -\bar{q}(t) \tag{6}$$

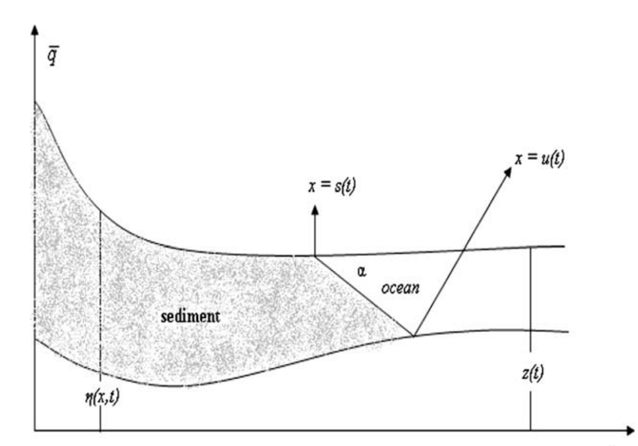

Fig. 1. A cross-section of sedimentary ocean basin.

and

$$\eta(s,t) = 0 \tag{7}$$

The additional conditions on the moving interface are

$$-\nu\frac{\partial\eta}{\partial x}\bigg|_{x=s(t)} = \gamma s\frac{ds}{dt} \tag{8}$$

and

$$s(0) = 0 \tag{9}$$

where $\alpha(u-s)=\frac{\alpha\beta s}{\alpha-\beta} = \gamma s$ and γ is a constant.

3. Solution of the problem

Describing the space variable x by $x_i = ik$, $(i = 0,1,2,\ldots,n+1)$ and using central difference, Eqs. (5)–(7) can be written in vector–matrix form as

$$\frac{d\eta}{dt} = \mathbf{A}\eta + \mathbf{B} \tag{10}$$

where

$$\eta = [\eta_1, \eta_2, \ldots, \eta_n]^T$$

$$\mathbf{B} = \frac{\nu}{k^2}\left[\frac{k\bar{q}}{\nu}, 0, \ldots, 0\right]^T$$

$$\mathbf{A} = \frac{\nu}{k^2}\begin{bmatrix} -1 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & \ldots & 0 & 0 & 0 \\ .. & & & .. & & .. & & \\ .. & & & .. & & .. & & \\ 0 & 0 & 0 & 0 & \ldots & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 & -2 \end{bmatrix}_{n\times n}$$

Integrating Eq. (10) and using the initial condition $\eta(0) = 0$, gives

$$\eta(t) = \mathbf{A}\int_0^t \eta(y)dy + \mathbf{B}\int_0^t 1.dy \tag{11}$$

The approximation of $\eta(t)$ by Chebyshev series gives:

$$\eta(t) = \mathbf{D}F(t) \tag{12}$$

$$1 = \mathbf{E}F(t) \tag{13}$$

where

$$\mathbf{D} = \begin{bmatrix} d_{11} & d_{12} & \ldots & d_{1m} \\ d_{21} & d_{22} & \ldots & d_{2m} \\ .. & .. & \ldots & .. \\ .. & .. & \ldots & .. \\ d_{n1} & d_{n2} & \ldots & d_{nm} \end{bmatrix}$$

$$E = [1, 0, 0, 0\ldots, 0]_{1\times m}$$

Table 1
The exact value, numerical value, absolute error and relative error at $\alpha = 1.7$.

β	S	Exact value t_A	Numerical value t_N	Absolute error $\|t_A - t_N\|$	Relative error $\|t_A - t_N\|/t_A$
0.5	0.1	0.005110	0.0035	0.001610	0.31506
	0.2	0.021768	0.0192	0.001568	0.11790
	0.3	0.048978	0.0475	0.001478	0.03020
	0.4	0.087100	0.0862	0.000900	0.01030
	0.5	0.136051	0.1377	0.000649	0.00477
	0.6	0.195910	0.1906	0.000531	0.00271
	0.7	0.276670	0.2762	0.000470	0.00169
1.0	0.1	0.014400	0.0096	0.00484	0.33333
	0.2	0.056522	0.0529	0.00362	0.06408
	0.3	0.129424	0.1271	0.00232	0.01796
	0.4	0.230087	0.2279	0.00219	0.00951
	0.5	0.359500	0.3575	0.00198	0.00551
	0.6	0.517700	0.5160	0.00167	0.00322
	0.7	0.704640	0.7032	0.00144	0.00204
1.5	0.1	0.066119	0.0429	0.02322	0.35117
	0.2	0.264750	0.2445	0.02025	0.07649
	0.3	0.595680	0.5841	0.01158	0.01944
	0.4	1.058990	1.0495	0.00951	0.00898
	0.5	1.654670	1.6528	0.00190	0.00115
	0.6	2.382720	2.3813	0.00145	0.00061
	0.7	3.243150	3.2423	0.00084	0.00026

Table 2
The exact value, numerical value, absolute error and relative error at $\beta = 0.5$.

α	S	Exact value t_A	Numerical value t_N	Absolute error $\|t_A - t_N\|$	Relative error $\|t_A - t_N\|/t_A$
1.0	0.1	0.007011	0.0050	0.002011	0.28683
	0.2	0.028043	0.0261	0.001943	0.06929
	0.3	0.063097	0.0618	0.000297	0.00471
	0.4	0.112173	0.1119	0.000273	0.00243
	0.5	0.175270	0.1751	0.000170	0.00097
	0.6	0.252389	0.2523	0.000089	0.00035
	0.7	0.343529	0.3435	0.000029	0.00008
1.4	0.1	0.005442	0.0037	0.001174	0.32010
	0.2	0.026101	0.0246	0.001501	0.07220
	0.3	0.052383	0.0509	0.001483	0.02831
	0.4	0.093125	0.0918	0.001325	0.01422
	0.5	0.145508	0.1445	0.001008	0.00682
	0.6	0.204531	0.2043	0.000176	0.00086
	0.7	0.286052	0.2860	0.000052	0.00018
1.7	0.1	0.005442	0.0035	0.001942	0.35680
	0.2	0.021768	0.0192	0.002568	0.11797
	0.3	0.048980	0.0475	0.001480	0.03021
	0.4	0.087075	0.0862	0.000875	0.01004
	0.5	0.136055	0.1358	0.000255	0.00187
	0.6	0.195920	0.1958	0.000120	0.00061
	0.7	0.266670	0.2666	0.000007	0.00026

and

$$F = [f_0, f_1, f_2, \ldots, f_{m-1}]^T_{1\times m}$$

f_j is Chebyshev polynomial of second kind such that

$$\begin{aligned} f_0 &= 1 \\ f_1 &= 2 - 4(t/\tau) \\ f_2 &= 3 - 16(t/\tau) + 16(t/\tau)^2 \\ &\vdots \\ &\vdots \\ f_{j+1} &= 2[1 - 2(t/\tau)]f_{j'} - f_{j-1} \end{aligned}$$

Moreover, integration of the Chebyshev vector gives

$$\int_0^t F(y)dy = \mathbf{P}F(t) \tag{14}$$

where $\mathbf{P}$ is the operational matrix of integration and

$$\mathbf{P} = \tau \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ \frac{3}{8} & 0 & -\frac{1}{8} & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{12} & 0 & -\frac{1}{12} & 0 & \ldots & 0 & 0 & 0 & 0 \\ . & . & . & . & . & \ldots & . & . & . & . \\ . & . & . & . & . & \ldots & . & . & . & . \\ \frac{1}{2(m-1)} & 0 & 0 & 0 & 0 & \ldots & 0 & \frac{1}{4(m-1)} & 0 & -\frac{1}{4(m-1)} \\ \frac{1}{2m} & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & \frac{1}{4m} & 0 \end{bmatrix}_{m\times m}$$

Substituting (12) and (13) in Eq. (11) and using (14), we obtained

$$\mathbf{D}F = \mathbf{ADP}F + \mathbf{BEP}F$$

Since the Chebyshev polynomial are independent, equating the coefficients of $F(t)$ gives the following set of Linear algebraic equations

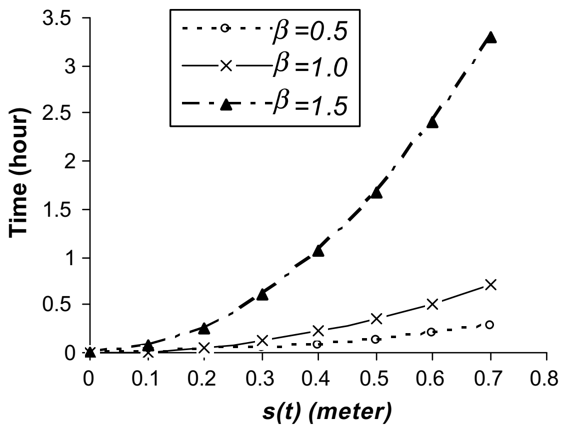

Fig. 2. Dependence of interface position on time for different basement slope.

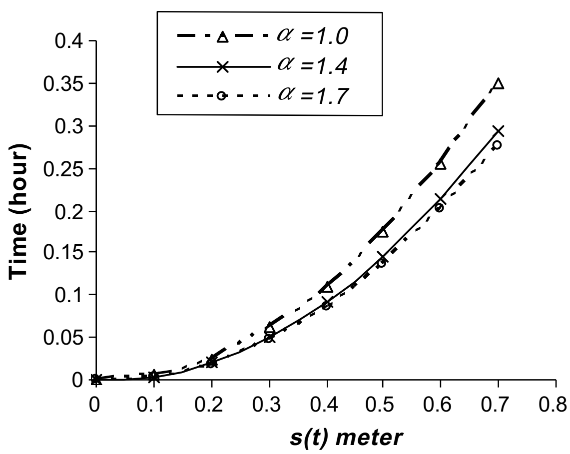

Fig. 3. Dependence of interface location on time for different slope of off-shore sediment wedge.

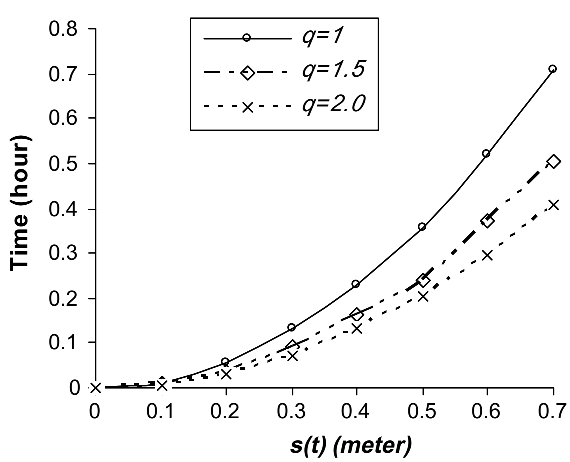

Fig. 4. Dependence of shoreline position on time for different sediment line flux.

$$\mathbf{D} - \mathbf{ADP} = \mathbf{BEP} \quad (15)$$

Now, we look for the time in which the interface moves a distance s. The region $(0,s)$ is divided into $n+1$ equal parts or sub regions. Replacing the space derivative by using backward operator in the interface condition (6) and integrating it with $s(0) = 0$, we obtain

$$S^2(t) = \frac{2v}{\gamma k}\left[\int_0^t \eta_i(y)dy\right], \quad (16)$$

where $i = 0, 1, \ldots, n$

By assuming a fixed $\tau > t$, the elements of the matrix **D** whose order is considered as $n \times 3$ (considering $m = 3$) are computed from equation (15). Replacing η_i by $\sum_{j=1}^{3} d_{ij} f_{j-1}$ and putting $\tau = t$ in Eq. (16), we obtain the first approximation as

$$t_1 = \frac{s^2\gamma k}{2v\left[d_{i1} + \frac{1}{3}d_{i3}\right]} \quad (17)$$

which gives the required time in which the shoreline moves at a distance s.

Again choosing τ the same as above calculated t_1, the new elements of matrix **D** are calculated from (15). These new elements of matrix **D** are used in Eq. (17) to evaluate new estimated t_2. This iterative process is continued until the difference between two successive values of t becomes smaller than a prescribed accuracy.

4. Numerical results and discussion

In this section, numerical results of time for various values of shoreline position (s) are calculated for different slope of basement (β) and off-shore sediment (α). All the numerical computations have been done for fixed line flux $\bar{q} = 1m^3/mt$, $v = 2m^2/t$ and are carried out using Mathematica software. Table 1 shows the approximate numerical solution t_N, exact solution t_A at $\alpha = 1.7$, $\beta = 0.5$, 1.0, 1.5 and the absolute error and the relative error between them. Table 2 shows the t_N, t_A at $\beta = 0.5$, $\alpha = 1.0$, 1.4, 1.7 and the absolute error and the relative error between them. From the Tables we observed that our approximate numerical method is in good agreement with the exact solution. Not surprisingly, the accuracy of the result can be improved by increasing the value of m.

It is seen from Fig. 2 that for a fixed value of slope of off-shore sediment wedge ($\alpha = 1.7$), if the value of β increases ($\beta = 0.5, 1.0, 1.5$), the movement of shoreline position decreases. In this case the sedimentation process becomes slow. The physical interpretation of increasing β implies that the sediments will be deposited towards the land side which causes the increase of the thickness of earlier sediments. As a consequence of this there will be least shifting of the contact point towards the land side and sedimentation process will be slower. Further for higher values of β (i.e., when β is closed to α), the downward warping of the sedimentary strata becomes very prominent and eventually the point of contact moves slowly towards the land and as a result sedimentation process becomes more slow.

Fig. 3 depicts that for a fixed value of basement slope ($\beta = 0.5$), the increase in α ($\alpha = 1.0, 1.4, 1.7$) represents more sediments will be deposited near the contact point which causes the advancement of the shoreline position towards the sea and as a result the sedimentation process becomes fast.

It is also seen from Fig. 4 that for fixed values of $\alpha = 1.7$, $\beta = 1.0$, $v = 2.0$, if the sediment line flux $\bar{q}$ increases ($\bar{q} = 1.0, 1.5, 2.0$), the movement of the shoreline position increases towards sea side with formation of inclined strata along the off-shore sediment wedge.

5. Conclusion

The utility of this numerical technique can be attributed to its simplistic approach in seeking the solution of the moving-boundary problem. In view of rapid convergence of the Chebyshev series of second kind, only a few terms of the series are needed to give satisfactory results [11]. It is often necessary to take moderately small step size to avoid undesirable numerical oscillation. The procedure as described in the present study will be applicable to linear and nonlinear moving-boundary problems.

Acknowledgements

The authors express their sincere thanks to the referees for their valuable suggestions for the improvement of the paper.

References

[1] J. Crank, Free and Moving Boundary Problem, Cambridge University Press, Oxford, 1984.
[2] H.S. Carslaw, J.C. Jaeger, Conduction of Heat in Solids, Oxford University Press, Oxford, UK, 1987.
[3] J.B. Swenson, V.R. Voller, C. Paola, G. Parker, J.G. Marr, Fluvio-deltaic sedimentation: a generalized Stefan problem, Eur. J. Appl. Maths. 11 (2000) 433–452.
[4] V.R. Voller, J.B. Swenson, C. Paola, An analytic solution for a Stefan problem with variable latent heat, Int. J. Heat Mass Transfer 47 (2004) 5387–5390.
[5] V.R. Voller, C.R. Swaminathan, B.G. Thomas, Fixed grid techniques for phase change problems, a review, Int. J. Num. Meth. Eng. 30 (1990) 875–898.
[6] V.R. Voller, An overview of numerical methods for phase change problems, Adv. Num. Heat Transfer 1 (1996) 341–375.
[7] D.R. Lynch, K. O'Neill, Continuously deforming finite elements for the solution of parabolic problems with and without phase changes, Int. J. Num. Meth. Eng. 17 (1981) 81–96.
[8] D.R. Lynch, Unified approach to simulation on deforming elements with application to phase change problems, J. Comp. Phys. 47 (1982) 387–411.
[9] C. Paola, P.L. Heller, C.L. Angevine, The large-scale dynamics of grain-size variation in alluvial basins, Basin Res. 4 (1992) 73–90.
[10] H.S. Uday Kumar, R. Mittal, Wei Shyy, Computation of solid–liquid phase fronts in the sharp interface limit on fixed grid, J. Comp. Phys. 153 (1999) 535–574.
[11] C.C. Liu, Y.P. Shih, Analysis and parameter identification of linear systems via Chebyshev polynomials of second kind, Int. J. Syst. Sci. 16 (1985) 753–759.
[12] H. Capart, M. Bellal, D.-L. Young, Self-similar evolution of semi-infinite alluvial channels with moving boundaries, J. Sedimentary Res. 77 (2007) 13–22.
[13] V.R. Voller, J.B. Swenson, W. Kim, C. Paola, An enthalpy method for moving boundary problems on the Earths' surface, Int. J. Num. Heat Fluid Flow 16 (2006) 641–654.